

# Cusped hyperbolic 3-manifolds: canonically CAT(0) with CAT(0) spines

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## Abstract

We prove that every finite-volume hyperbolic 3-manifold  $M^3$  with  $p \geq 1$  cusps admits a canonical, complete, piecewise Euclidean CAT(0) metric, with a canonical projection to a CAT(0) spine  $K_M^*$ . Moreover:

- (a) The universal cover of  $M^3$  endowed with the CAT(0) metric is a union of Euclidean half-spaces, glued together by identifying Euclidean polygons in their bounding planes by pairwise isometry;
- (b) Each cusp of  $M^3$  in the CAT(0) metric is a non-singular metric product  $\mathbb{E}_i^* \times [1, \infty)$ , where  $\{\mathbb{E}_i^*\}_{i=1}^p$  is a set of Euclidean cusp tori, with  $\mathbb{E}_i^*$  having the canonical shape associated with the  $i$ th cusp;
- (c) Metric singularities are concentrated on the 1-skeleton of  $K_M^*$  with cone angle  $k\pi$  on any edge of degree  $k$ . The CAT(0) 2-complex  $K_M^*$  is constructed canonically from Euclidean polygons  $P_{i,j}^e$ , which reassemble to create  $\{\mathbb{E}_i^*\}_{i=1}^p$ ;
- (d) There is a canonical 1-parameter metric deformation, through piecewise-constant-curvature complete metrics, from the hyperbolic metric with limit the piecewise Euclidean one (facilitated by a simple application of Pythagorus' Theorem);
- (e) The hyperbolic metric on  $M$  can be reconstructed from a finite set of points  $p_{i,j}$  on the tori  $\mathbb{E}_i^*$ , weighted by real numbers  $w_{i,j} \in (0, 1)$ .

Our CAT(0) construction can be considered ‘dual’ to that of Epstein and Penner, but uses much simpler arguments, directly and canonically based on Ford domains. Epstein and Penner’s metrics, parametrized by a choice  $\mathcal{T}$  of disjoint cusp horotori, gives rise to incomplete piecewise Euclidean metrics with singularities in cusps. To each such choice  $\mathcal{T}$ , we also construct a complete CAT(0) metric of the above form, with CAT(0) spine  $K_{\mathcal{T}}$ . This CAT(0) metric structure is already visible via both Weeks’ Snappea program, and its recent manifestation SnapPy by Culler and Dunfield, although its existence has not previously been observed.

Our construction also generalizes to finite-volume  $p$ -cusped  $n$ -manifolds  $W^n$ , to endow each with a complete piecewise-Euclidean CAT(0) metric with non-singular product end structures, whose singularities are concentrated in codimension 2: such  $W^n$  deformation retract to a natural spine, which is CAT(0) as a manifestation of polar duality of ideal hyperbolic polytopes.

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# 1 Introduction

Spaces of constant curvature play a fundamental role in pure mathematics, since their advent as solutions to the conceptual problem of the independence of Euclid's 5th Postulate. Relationships between geometry, complex analysis and number theory, and between discrete and continuous representations of mathematical objects, continue to be of profound significance. We demonstrate a simple interplay of the combinatorics and smooth structure of non-compact hyperbolic space-forms, geometrically dual to and inspired by Epstein and Penner's canonical decompositions [18].

Generically, a closed hyperbolic manifold  $M$  does not have a natural distinguished finite set of points by which to create a combinatorial structure: any given finite set of points enables the construction of a Delauney decomposition, or dually, a Dirichlet/Voronoi decomposition of  $M$  into hyperbolic cells, as demonstrated by Näätänen-Penner [27] in two dimensions, and more generally by Charney-Davis-Moussong [13], using the Minkowski model for hyperbolic geometry. The results of [13] also pertain to non-compact hyperbolic manifolds, but are less canonical in that case, requiring a careful choice of infinitely many points by which to create a locally-finite cell decomposition into compact cells.

In these constructions, it can be shown that hyperbolic cells can be replaced by Euclidean ones:  $M$  admits piecewise Euclidean CAT(0) structures, in the sense of Gromov. For dimension greater than 2, this utilises Rivin's description of ideal hyperbolic polyhedra in 3-space, using polar duality, and its generalisation to higher dimensions by Charney-Davis [12].

The results of [13, 27] were motivated by Epstein-Penner's [18] canonical decomposition of a cusped hyperbolic manifold  $M$  of finite volume into ideal hyperbolic polytopes, using their convex hull construction in Minkowski space: this construction exploits the distinguished finite set of 'ideal points' of  $M$  corresponding to cusps. The Epstein-Penner piecewise Euclidean metrics arise naturally by replacing hyperbolic polytopes with Euclidean ones, but are incomplete, and have singular set intersecting cusps.

Epstein-Penner's canonical decomposition is essentially a Delauney decomposition, based on the ideal points of  $M$ . Dual to any Delauney decomposition is a Voronoi-Dirichlet decomposition, and Epstein and Penner show that their decomposition of  $M$  into hyperbolic polytopes is naturally dual to the classical Ford decomposition, which is traditionally defined using isometric circles in the upper-half-space model for hyperbolic space. Our proof is based on the geometry of Ford domains, viewed – as heuristically described in [18] – as arising by the collision of expanding horospheres, and is thus based on a Voronoi-Dirichlet construction.

Thus, of all four CAT(0) structures defined on a cusped hyperbolic manifold  $M$ , arising respectively from the Delauney, Voronoi-Dirichlet, Epstein-Penner, and Ford decompositions of  $M$  into hyperbolic pieces, the latter two are most natural, but only our decomposition gives a complete metric, a CAT(0) spine, and singularity-free cusps.

The proof of our main theorem generalises to any dimension, and so we concentrate on dimension 3 for the purposes of illustration and exposition, and the contextual significance of the conclusion: for 2-dimensional analogues, with applications to Riemann surfaces and moduli thereof, we refer to the work of Bowditch and Bowditch-Epstein [7, 8]. For higher dimensions, the proof is essentially identical, given the Charney-Davis results on polar duality [12].

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### 3 Preliminaries and notation

We denote hyperbolic 3-space by  $\mathbb{H}^3$ , and by  $\bar{\mathbb{H}}^3$  its compactification obtained by adding ideal points. These constitute the sphere at infinity,  $S_\infty^2 = \bar{\mathbb{H}}^3 - \mathbb{H}^3$ . The upper-half space model  $\mathbb{UH}^3$  for  $\mathbb{H}^3$  has underlying set

$$\mathbb{UH}^3 := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} = \mathbb{R}_+^3$$

and sphere at infinity represented as

$$S_\infty^2 = \mathbb{R}_0^2 \cup \infty := \{(x, y, 0) \in \mathbb{R}^3\} \cup \infty,$$

with  $\mathbb{R}_0^2$  inheriting a Euclidean metric, up to similarity: for any  $p \in \mathbb{R}_0^2$ , any dilation of  $\mathbb{R}^3$  centered at  $p$  gives a hyperbolic isometry, as does any translation fixing  $\mathbb{R}_0^2$  setwise.

In  $\mathbb{UH}^3$ , any horosphere appears as either a horizontal Euclidean plane,

$$HoP_a := \{(x, y, a) \in \mathbb{R}_+^3\},$$

or as a Euclidean sphere  $HoS_{p,d}$  of diameter  $d$ , tangent to  $\mathbb{R}_0^2$  at  $p$ , which is deleted.

Similarly, in  $\mathbb{UH}^3$ , a hyperbolic plane appears as either a vertical Euclidean plane,

$$\mathbb{Hy}P_{a,b,c}^2 := \{(x, y, z) \in \mathbb{R}_+^3 \mid ax + by + c = 0\} \cup \infty,$$

after deleting its *circle-at-infinity*  $S_{a,b,c}^1 := \{(x, y, 0) \in \mathbb{R}_0^2 \mid ax + by + c = 0\} \cup \infty$ ; or as a Euclidean hemisphere  $\mathbb{Hy}S_{p,r}^2$  of radius  $r$ , centered on  $\mathbb{R}_0^2$  at  $p$ , with the equatorial boundary circle  $S_{p,r}^1 \subset \mathbb{R}_0^2$ , its circle-at-infinity, deleted.

If  $P^h \subset \mathbb{Hy}S_{p,r}^2$  is any compact, *hyperbolic*, polygon, its orthogonal projection to  $\mathbb{R}_0^2$  is a compact, Euclidean polygon  $P^e$ , with respect to the standard Euclidean metric on  $\mathbb{R}_0^2$ . Each edge of  $P^e$  determines a vertical plane in  $\mathbb{R}_+^3$ , and hence a hyperbolic plane in  $\mathbb{UH}^3$  intersecting  $\mathbb{H}_{p,r}^2$  in a hyperbolic geodesic containing a geodesic boundary segment of  $P^h$ .

**Definition.** A *label* for  $P$  is a pair  $(p, r) \in \mathbb{R}_0^2 \times \mathbb{R}_+ \cong \mathbb{C} \times \mathbb{R}_+$ . (We do not associate a label to polygons lying in vertical hyperbolic planes.)

A non-compact complete hyperbolic 3-manifold  $M^3$  of finite volume  $M^3$  has  $p$  cusps for some  $p \geq 1$ , and decomposes [37, 38] as  $M^3 = M^{thick} \cup \{\mathcal{C}_i\}_{i=1}^p$ , with compact ‘thick’ part  $M^{thick}$  having as complement a disjoint union of  $p$  cusps  $\mathcal{C}_i$ ,  $i = 1, \dots, p$ , each topologically a product  $T^2 \times (1, \infty)$  of tori. In the hyperbolic metric, each torus  $T^2 \times \{t\}$  inherits a Euclidean metric, whose scale shrinks exponentially as  $t \rightarrow \infty$  at unit speed. Accordingly, there is canonically associated to  $M^3$  a set  $\{\mathbb{E}c_i\}_{i=1}^p$  of elliptic curves, with  $\mathbb{E}c_i$  associated to the  $i$ th cusp  $\mathcal{C}_i$ .

A *cuspidal curve* for  $M^3$  is any elliptic curve  $\mathbb{E}c_i$  associated to a cusp of  $M^3$ . A *cuspidal torus*, or *horotorus* in  $M^3$  is the image of any *Euclidean torus*  $\mathbb{E}t$  isometrically embedded in some cusp of  $M^3$ .

Given any set  $\{\mathbb{E}c_i\}_{i=1}^p$  of elliptic curves, since each curve  $\mathbb{E}c_i$  admits a unique flat Euclidean metric, up to scale, we obtain a set of Euclidean tori  $\{\mathbb{E}t_i\}_{i=1}^p$  by independently specifying a scale for each. A priori, there is no specified scale for each cusp elliptic curve: specifying a scale amounts to choosing a cusp torus, and each cusp determines a least upper bound for the possible size of any cusp torus, and hence determines a *maximal cuspidal torus*  $\mathbb{E}^*t_i$ ,  $i = 1, \dots, p$ . Each  $\mathbb{E}^*t_i$  has a non-empty finite set of self-tangencies, and thus determines a finite set of points on the corresponding elliptic curve.

By Marden and Prasad’s generalization [26, 31] of Mostow rigidity, there is a unique (up to conjugation) representation  $\rho : \pi_1(M^3) \rightarrow \Gamma = \rho(\pi_1(M^3)) \subset PSL_2(\mathbb{C}) \cong Isom_+(\mathbb{H}^3)$ , with  $M \cong \mathbb{H}^3/\Gamma$ . Thus  $\Gamma$  naturally acts on  $\mathbb{UH}^3$ , by Möbius transformations on the Riemann sphere  $\hat{S}^2 = \mathbb{C} \cup \infty \cong \mathbb{R}_0^2 \cup \infty$ , and  $\Gamma$  acts on  $\bar{H}^3$  with a dense set  $\mathcal{P}_\Gamma \subset S_\infty^2$  of parabolic fixed points, falling into  $p$  distinct orbits corresponding to the  $p$  cusps of  $M^3$ .

The preimage in  $\mathbb{H}^3$  of any cusp torus is a disjoint set of horospheres with inherited Euclidean metric. Such metrics can be seen algebraically in the Lorentzian model, and are visually natural in the upper-half space model for horospheres  $Hop_a$  centered at  $\infty \in S_\infty^2 = \infty \cup \mathbb{R}_0^2$ . Accordingly the upper-half-space models we use to describe  $M^3$  will have  $\infty$  as a parabolic fixed point for  $\Gamma$ : For  $p_i \in S_\infty^2$  a parabolic fixed point corresponding to the  $i$ th cusp, we conjugate  $\Gamma$  to  $\Gamma_i$  so that  $p_i$  is at  $\infty_i := \infty$  in  $\mathbb{UH}_i^3 := \mathbb{UH}^3$ ; thus the stabilizer of  $\infty_i$  is a  $\mathbb{Z} \oplus \mathbb{Z}$  subgroup of  $\Gamma_i$ . When  $p > 1$ , although  $\Gamma_i$  is conjugate to  $\Gamma_j$  in  $PSL(2, \mathbb{C})$ , there is no element of  $\Gamma_i$  sending  $\infty$  to any parabolic fixed point corresponding to a distinct cusp  $\mathcal{C}_j$  of  $M^3$ , since the images of  $\infty$  under  $\Gamma_i$  constitute a single orbit of parabolic fixed points.

**Definition.** Denote by  $\mathbb{UH}_i^3$  the  $i$ th upper-half-space on which  $\Gamma_i$  acts. When a horotorus  $\mathbb{Et}_i \subset \mathcal{C}_i$  is specified, we will generally assume that we have conjugated  $\Gamma$  so that the horosphere  $HoP_1$  projects to  $\mathbb{Et}_i$  under the action of  $\Gamma_i$  on  $\mathbb{UH}_i^3$ . The metric on the cusp torus  $\mathbb{Et}_i$  is determined by the action of the parabolic subgroup stabilizing  $\infty$ , since the induced Euclidean metric on  $HoP_1 = \{(x, y, 1)\} \cong \{(x, y)\} = \mathbb{R}^2$  is the standard Euclidean metric.

## 4 Statement of results

**Theorem 1.** *Suppose  $M^3$  is a non-compact, connected 3-manifold admitting a complete hyperbolic metric of finite-volume with  $p \geq 1$ -cusps. Then*

- *$M^3$  admits a complete piecewise-Euclidean  $CAT(0)$  metric, with singular set concentrated on a finite connected graph; all edge cone angles are of form  $k\pi$ ,  $3 \leq k \in \mathbb{Z}$ .*
- *Each hyperbolic cusp  $\mathcal{C}_i$ ,  $i = 1, \dots, p$ , canonically determines an elliptic curve  $\mathbb{Ec}_i$  (a Euclidean similarity class of a closed Euclidean torus): there is a consistent choice  $\mathbb{Et}_i^*$  of a representative Euclidean torus from each class, inducing the  $CAT(0)$  metric on  $M^3$ , with each cusp the Euclidean metric product  $\mathbb{Et}_i^* \times [1, \infty)$ .*
- *Each  $\mathbb{Et}_i^*$  is a union of convex Euclidean polygons  $P_{i,j}^e$ : the  $CAT(0)$  metric on  $M^3$  arises as the quotient space of  $\coprod \mathbb{Et}_i^* \times [1, \infty)$  by Euclidean isometric identification of pairs of polytopes  $P_{i,j}^e \times \{1\}$ .*
- *The piecewise-Euclidean 2-complex  $K$  obtained by pairwise identification of polytopes  $P_{i,j}^e \times \{1\}$  is a spine for  $M^3$ , and is  $CAT(0)$ .*
- *There is canonical deformation between the unique hyperbolic and  $CAT(0)$  metrics, via a natural manifestation of Pythagoras' Theorem.*
- *The decomposition of Euclidean tori into polygons  $P_{i,j}^e$  is determined by a canonical finite set of weighted points  $p_{i,j} \in \mathbb{Et}_i^* \times (0, 1)$ . The hyperbolic metric on  $M^3$  can be reconstructed from the data  $\{p_{i,j}\}$ .*
- *The universal cover of  $M^3$  with  $CAT(0)$  metric is a union of Euclidean half-spaces, corresponding to hyperbolic horoballs, glued together by pairwise isometry of Euclidean polygons forming tessellations of their bounding Euclidean planes.*

## 5 Ford domains

Epstein and Penner define their canonical decomposition using the Lorentzian model: traditional Ford domains are naturally seen in the upper-half space model, since classically their construction is via isometric circles in  $\mathbb{R}_0^2$ . In the 1-cusped case, Epstein and Penner formalise the heuristic ‘bumping locus’ construction by expanding horospheres in  $\mathbb{UH}^3$  [18], showing that their canonical decomposition of  $M^3$  into ideal polyhedra is naturally dual to the Ford complex: we describe ‘horospherical bumping’ for  $p \geq 1$  in more detail, working equivariantly in  $\mathbb{UH}^3$  as universal cover of  $M^3$ :

Take any disjoint union  $\mathcal{T} = \{\mathbb{E}t_i\}_{i=1}^p \subset M^3$  of cusp tori, one for each cusp. These lift to a union  $\mathcal{H} = \{H_p\}$  of horospheres centered at parabolic fixed points  $p \in S_\infty^2$ , equivariant with respect to the action of  $\Gamma$ , and determine a set of disjoint open horoballs  $\mathcal{B} = \{B_p\}$ . Expand each  $H_p$  at unit speed, allowing them to ‘flatten’ against each other, creating a locally finite piecewise geodesic 2-complex  $K_{\mathcal{H}}$ .

**Definition.** We call this ‘bumping locus’  $K_{\mathcal{H}}$  the *Ford complex* for  $\Gamma$  determined by  $\mathcal{H}$ . Its projection  $K_{\mathcal{T}} = K_{\mathcal{H}}/\Gamma$  is a 2-complex in  $M^3$ , which is the *Ford spine* for  $M^3$  determined by  $\mathcal{T}$ . There is a strong deformation retraction from  $M^3$  to  $K_{\mathcal{T}}$ .

Viewed from  $\infty_i$  for  $\mathbb{U}\mathbb{H}_i^3$ , the visible part  $K_{\mathcal{H}}^{\infty_i}$  of  $K_{\mathcal{H}}$  is a locally finite piecewise geodesic 2-complex constructed from compact hyperbolic polygons  $P_{i,j}^h$ , projecting to Euclidean polygons  $P_{i,j}^e$  tessellating  $HoP_1$  (or, equivalently,  $\mathbb{R}_0^2$ ).

The Ford complex is created by numerous expanding geodesic discs in such hyperplanes, which in turn intersect each other creating the 1-skeleton: viewed from  $\infty_i$ , the Euclidean projections of these discs expand until they encounter other expanding discs, at which stage their boundary circles also ‘flatten’ against each other creating the straight boundary-edges of Euclidean polygons  $P_{i,j}^e$ . However, these expanding Euclidean discs do not expand at constant rate: we discuss this later.

**Definition.** By abuse of language, we call the closure of the complementary component of  $K_{\mathcal{H}}^{\infty_i}$  in  $\mathbb{U}\mathbb{H}_i^3$  containing the horoball  $B_\infty$  a *Ford ball* for  $\Gamma_i$ , denoted by  $FB_{\mathcal{H},i}$ : this is non-compact, has boundary  $K_{\mathcal{H}}^{\infty_i}$  with infinitely many faces, and is the analogue of a Dirichlet polyhedron, with center at  $\infty_i$ .

The pair  $(\Gamma, \mathcal{H})$  determines an equivariant tessellation of  $\mathbb{H}^3$  by copies of Ford balls  $FB_{\mathcal{H},i}$ . This tessellation is equivalently created by uniformly expanding all horoballs in  $\mathcal{B}$ , allowing them to flatten against each other. Each  $FB_{\mathcal{H},i}$  is stabilized by a  $\mathbb{Z} \oplus \mathbb{Z}$  subgroup, and projects to a neighbourhood, denoted  $FC_{\mathcal{K},i}$ , of the cusp  $\mathcal{C}_i$ . We call these *Ford cusps*: these are the closures in  $M^3$  of the complementary components to  $K_{\mathcal{T}}$ , and each contains a horotorus  $\mathbb{E}t_i$  naturally decomposed as a union of Euclidean polygons  $P_{i,j}^e$ .

**Proposition 1.**  $\mathbb{H}^3$  is obtained from the disjoint union of Ford balls, by isometric pairwise identification of hyperbolic polygons  $P_{i,j}^h$  in their boundaries.

**Definition.** Define a *hyperbolic Ford polytope*  $FP_{\mathcal{H},i,j}^h$  for  $\Gamma_i$  to be the closure in  $\mathbb{U}\mathbb{H}_i^3$  of the region vertically above a hyperbolic polygon  $P_{i,j}^h$ ; similarly, define a *Euclidean Ford polytope*  $FP_{\mathcal{H},i,j}^e$  to be the closure in  $\mathbb{R}_i^3$  (underlying  $\mathbb{U}\mathbb{H}_i^3$ ) of the region vertically above a Euclidean polygon  $P_{i,j}^e \subset HoP_1$ . Both can be construed as ‘cones of polygons to infinity’. Each  $FB_{\mathcal{H},i}$  is a union of hyperbolic Ford polytopes, equivariantly with respect to the action of  $\Gamma_i$ . Thus

**Proposition 2.**  $M^3$  is obtained from the disjoint union of hyperbolic Ford polytopes, by pairwise isometric identification of hyperbolic polygons in their boundaries.

The finitely-many boundary faces of a Ford polytope consist of some hyperbolic  $m$ -gon  $P_{i,j}^h$ , together with  $m$  non-compact hyperbolic triangles with exactly one ideal vertex.

Dual to the Ford complex is a decomposition of  $\mathbb{U}\mathbb{H}^3$  into ideal polyhedra, which generically are simplices: the ideal cell dual to a given 0-cell  $x$  of the Ford complex is the convex hull of the set of parabolic points determined by the closest equidistant horospheres. These polytopes form the Epstein–Penner *canonical decomposition* [18], which is unique when  $p = 1$ , but otherwise admits a parameter space of real dimension  $(p - 1)$  corresponding to the  $p$  choices of disjoint horotori, up to simultaneous rescaling. Akiyoshi has shown in [3] that a finite volume hyperbolic manifold with multiple cusps admits finitely many combinatorial types of canonical cell decompositions.

## 6 The distinguished Ford complex

The ‘dynamic’ view allows us to generalise, to an arbitrary  $p$ -cusped manifold, the heuristic described and made more precise in the 1-cusped case by Epstein–Penner [18]. When  $p = 1$ , all choices of embedded horotorus are equal after contraction or expansion, and the Ford complex forms from the instant any expanding horotorus first contacts itself, and thus no choice in the construction is possible: it is natural for the unique Ford complex to be seen as arising from expanding balloons flattening against each other, or via isometric spheres. An appropriate generalisation of a distinguished choice for a Ford complex when  $M^3$  has  $p > 1$  cusps is not immediately clear from this ‘balloon flattening’ perspective, since there is a  $(p - 1)$ -dimensional parameter space for spines arising from possible initial choices of disjoint cusp tori. Consider three distinct heuristic scenarios as expanding horotori encounter each other, with a view to adjusting an initial family of embedded horotori to create a more natural one:

**Flattening:** This is described above: locally, expanding horospheres flatten against each other. The expansion process stops when each point of each horosphere has encountered another;

**Immersed transition:** Instead of flattening against each other, allow horospheres to continue expanding, becoming immersed. Collectively the expanding horospheres eventually pass through all points in the complement of their horoball, and create quite complicated intersection patterns;

**Domination and submission:** Partition the set of cusps into two subsets, and declare one subset to be dominant, the other submissive. When two dominant horospheres meet, they flatten against each other; submissive horospheres are pushed back into their cusps by expanding dominant horospheres. In  $\mathbb{UH}_i^3$ , for  $i$  submissive, a horizontal horosphere eventually rises, supported by tangency with expanding dominant horospheres. Freeze the evolution at some instant, reverse the process by shrinking all immersed horotori back to disjointly embedded ones, and then allow all to expand again, but now all flattening against each other.

**Proposition 3.** *There is a distinguished 1-parameter family  $H^*$  of disjointly embedded cusp tori giving a corresponding unique Ford complex  $K_M^*$  for  $M^3$ , defined independently of  $p \geq 1$ , naturally generalising the case  $p = 1$ , by viewing the Ford complex as created by initially intersecting expanding horospheres.*

**Proof:** Consider expanding one arbitrarily chosen embedded horotorus  $\mathbb{Et}_i$ , ignoring all others, and allowing self-intersection rather than self-flattening. After finite time, the expanding torus  $\mathbb{Et}_i^t$  sweeps past all points of  $M^{thick}$ , leaving only points of cusps not yet encountered: the torus  $\mathbb{Et}_i^t$  must, by this stage, be immersed, not embedded: this is clear when  $p = 1$ ; and for  $p > 1$ , clear since the set of unencountered points of  $M^3$  is disconnected.

In keeping with the democratic philosophy of treating all cusps equally, we consider the set  $\{\mathbb{Et}_i^t\}$  of all maximal cusp tori. Collectively they form a non-transverse immersion of  $p$  Euclidean tori, and each of these can be independently shrunk at unit speed to become embedded. Doing this uniformly and simultaneously for each, we obtain a regular homotopy of  $p$  immersed Euclidean tori, which eventually becomes a family  $\mathcal{T}^*$  of embedded cusp tori (with 1-parameter set of choices by re-scaling uniformly). Allowing all these now-embedded cusp tori to expand again, we construct their collision locus  $K_M^* := K_{\mathcal{T}^*}$ .

**Definition.** The *distinguished* Ford spine  $K_M^*$  for  $M^3$  is defined to be  $K_M^* := K_{\mathcal{T}^*}$ , for any such embedded family of rescaled maximal tori.

The defining characteristic for  $K_M^*$  is arguably the most natural definition for distinguishing a family of cusp tori: others, perhaps less natural, can be defined using similar notions. For example, let  $\mathbb{Et}_i^{thick}$  denote the immersed torus obtained by expanding  $\mathbb{Et}_i^*$  until the first instant it has encountered each point of each other  $\mathbb{Et}_j^*$ . By this time, all points of  $M^{thick}$  have been encountered by the  $i$ th expanding cusp torus. Take the union  $\{\mathbb{Et}_i^{thick}\}_{i=1}^p$  of all these immersed tori, and uniformly and simultaneously shrink each backwards until each is embedded, and then allow the resulting embedded family  $\mathcal{T}_{thick}$  to expand to create  $K_M^{thick} := K_{\mathcal{T}_{thick}}$ , which is another natural choice for a distinguished family when  $p \geq 1$ .

## 7 Piecewise Euclidean structures: existence

We now define the Euclidean structure on  $M^3$ , arising from any Ford spine  $K_{\mathcal{T}}$ :

**Definition.** The piecewise Euclidean structure  $M_{\mathcal{T}}^3$  corresponding to  $\mathcal{T}$  is defined by replacing each polygon  $P_{i,j}^h$  by its projected Euclidean polygon  $P_{i,j}^e$ , replacing each Ford polytope  $FP_{\mathcal{H},i,j}^h$  by the Euclidean Ford polytope  $FP_{\mathcal{H},i,j}^e := P_{i,j}^e \times [1, \infty)$ , and replacing each Ford ball  $FB_{\mathcal{H},i}$  by the Euclidean half-space  $FB_{\mathcal{H},i}^e \cong \mathbb{R}_{\geq 1}^3 := \{(x, y, z) \mid z \geq 1\}$ , which is the union of Euclidean Ford polytopes.

Heuristically we vertically project those hyperbolic polygons in  $K_{\mathcal{H}}^{\infty i}$ , whose interior is visible from  $\infty_i$ , to the horizontal plane at height 1, and take the vertical half-infinite prism above their images. This is essentially coning each such hyperbolic polygon to the parabolic fixed point at infinity, intersecting with the horosphere at height 1, and ‘opening up’ each Ford polytope by no-longer-exponentially-shrinking Euclidean cusp geometry as we approach  $\infty$ . The labels assigned to polygons define an orthocenter for both hyperbolic and corresponding Euclidean polygons, as viewed from  $\infty$ .

It is clear that each Euclidean Ford ball  $FB_{\mathcal{H},i}^e$  has a geometric combinatorial structure equivariant under the  $\mathbb{Z} \oplus \mathbb{Z}$  action stabilizing  $\infty_i$ : we must show that the corresponding nonsingular quotients, which replace Ford cusps by Euclidean products, glue together to produce the singular piecewise Euclidean structure  $M_{\mathcal{T}}^3$ , whose universal cover is then a geometrically complete union of Euclidean half-spaces..

**Theorem 2.** *Suppose  $M^3$  is any non-compact, connected 3-manifold admitting a complete hyperbolic metric of finite-volume with  $p \geq 1$ -cusps, with any specified complete family of disjoint horotori  $\mathcal{T} = \{\mathbb{E}t_i\}$ .*

- *The metric structure  $M_{\mathcal{T}}^3$  is a complete piecewise-Euclidean metric, with singular set concentrated on a finite connected graph; all edge cone angles are of form  $k\pi$ ,  $3 \leq k \in \mathbb{Z}$ .*
- *Each  $\mathbb{E}t_i$  is decomposed naturally as a union of Euclidean polygons  $P_{i,j}^e$ : the piecewise Euclidean metric  $M_{\mathcal{T}}^3$  arises as the quotient space of  $\coprod \mathbb{E}t_i \times [1, \infty)$  by Euclidean isometric identification of pairs of polytopes  $P_{i,j}^e \times \{1\}$ .*
- *The piecewise-Euclidean 2-complex  $K_{\mathcal{T}}$  obtained by pairwise identification of polytopes  $P_{i,j}^e \times \{1\}$  is a piecewise Euclidean spine for  $M_{\mathcal{T}}^3$ .*

*Proof:* Each hyperbolic polygon  $P_{i,j}^h \in K_{\mathcal{H}}^{\infty i}$  is contained in a unique hyperplane  $HyS_{q,d}$ : we assign the label  $(q, d) \in \mathbb{R}_0^2 \times \mathbb{R}_+ \cong \mathbb{R}_+^3$  to  $P_{i,j}^h$ . Thus  $q$  is a parabolic fixed point for  $\Gamma_i$ , and  $P_{i,j}^h$  lies in the hyperbolic plane formed by  $H_{\infty_i}$  and  $\mathbb{H}_q$  flattening against each other. Let  $d = e^{-t}$ , where  $t$  denotes the time of initial tangency between these expanding horospheres since expansion began. Now  $q$  lies in some orbit corresponding to a cusp  $\mathcal{C}_k$ ,  $k \in \{1, \dots, p\}$ , and we consider the corresponding model  $\mathbb{UH}_k^3$ . In this picture, some polygon  $P_{k,s}^h$  in the orbit of  $P_{i,j}^h$ , and hence isometric to it by an orientation-reversing isometry (cf inversion in isometric spheres), is visible from  $\infty_k$ . These two hyperbolic polygons project to polygons in the boundary of Ford cusps in  $M$ , and are identified there by hyperbolic isometry gluing part of the boundaries of these cusps together.

Consider the label  $(q', d')$  for  $P_{k,s}^h$ . Since  $d' = e^{-t'}$  records the time of first tangency of expanding horospheres,  $t' = t$  and so  $d' = d$ : the corresponding hyperplanes in  $\mathbb{UH}_i^3, \mathbb{UH}_k^3$  appear with the same Euclidean diameter. Similarly, the points of  $P_{k,s}^h$  and  $P_{i,j}^h$  are created by circle expansion, and so can be put in correspondence: we may place both hyperplanes and polygons in the same  $\mathbb{UH}^3$ , and observe they can be made to coincide by orientation-reversing Euclidean congruence of  $\mathbb{R}_0^2$ .

Summarizing: if two hyperbolic polygons are identified by an element of  $\Gamma$ , the polygons each have the same height in their corresponding rescaled half space models. But a hyperbolic polygon with given label  $(q, *)$  in the upper half space model uniquely determines a Euclidean similarity class of Euclidean polygons by vertical projection; and specifying the height  $*$  uniquely determines the scale.

Now take a disjoint union  $\bigcup FB_{\mathcal{H},i}^e$  of a countably infinite number of copies of each Euclidean Ford ball/half-space. Then for each  $i$ ,  $\partial FB_{\mathcal{H},i}^e$  is a union of Euclidean polygons  $P_{i,j}^e$ , and to each we isometrically identify a

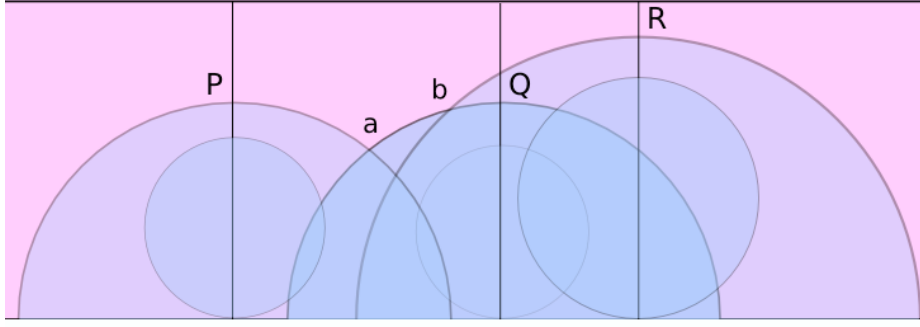


Figure 1: Three expanding horospheres  $HoS_*$ , and one descending  $HoP_*$ : three hyperplanes  $HyS_*$  with maximal points  $P, Q, R$ . The hyperbolic polygon  $ab$  is part of the Ford complex: its label is determined by the coordinates of  $Q$  in  $\mathbb{R}^3$ . Note that  $Q$  is invisible from  $\infty$ :  $ab$  is contained in a hyperbolic disc centered at  $Q$ , but does not itself contain  $Q$ .

corresponding  $FB_{\mathcal{H},k}^e$  by isometric identification with  $P_{k,s}^e$ . The resulting 3-complex is homeomorphic to  $\mathbb{UH}^3$ , and is metrically complete.

The Ford complex  $K_{\mathcal{H}}$  is replaced by, and is combinatorially equivalent to, its piecewise Euclidean counterpart  $K_{\mathcal{H}}^e$ , obtained from the disjoint union  $\bigcup \partial FB_{\mathcal{H},i}^e$  by pairwise isometric identification of all such Euclidean polygons  $P_{i,j}^e, P_{k,s}^e$ . Edges of  $K_{\mathcal{H}}^e$  correspond to those of  $K_{\mathcal{H}}$ , which have degree  $\geq 3$  (generically each edge has exactly 3 polygons incident with it). Since the Euclidean edges lie in the boundaries of half spaces, all edges have cone angle a multiple of  $\pi$  in  $M_{\mathcal{T}}^3$ .

Combinatorially, the structure is identical to that of the Ford complex, and so is equivariant with respect to the natural action of  $\pi_1(M^3)$ . The metric structure is equivariant, and so descends to define the metric  $M_{\mathcal{T}}^3$  with properties as stated in the theorem.

The structure we describe is no longer compatible with representations of  $\pi_1(M)$  in  $PSL(2, \mathbb{C})$ , for all  $M^3$ , simultaneously acting as isometries of the same space  $\mathbb{H}^3$ : Each piecewise Euclidean structure on  $M^3$  endows its universal cover, topologically  $\mathbb{R}^3$ , with piecewise Euclidean metrics which generally differ for different  $M^3$ , and different choices for  $\mathcal{T}$ .

## 8 Piecewise Euclidean structures are CAT(0)

For basic definitions for this section, we refer to [11, 33, 34, 12, 13]. In order to prove that the piecewise Euclidean structures we have defined on  $M_{\mathcal{T}}^3$  and its spine  $K_{\mathcal{T}}^e$  are CAT(0), we must argue that the link of each point is CAT(1): all geodesics in each piecewise spherical link should be of length at least  $2\pi$ . Such a piecewise spherical link is called *large*: there is a unique geodesic between any two points of distance less than  $\pi$ . The essence of the argument is that the Ford complex is geometrically dual to the canonical Epstein–Penner canonical decomposition into finite-volume ideal hyperbolic polyhedra, and that the links of vertices in the piecewise Euclidean structures  $K_{\mathcal{T}}^e$  and  $M_{\mathcal{T}}^3$  are essentially the polar duals of these hyperbolic polyhedra. Rivin [34] showed that the polar dual of a convex ideal hyperbolic polyhedron is large in dimension 3. This result was generalized by Charney and Davis [12] to higher dimensions, and accordingly we adapt some of their notation so that relevant parts of their description are clearer in the present context.

We must consider the link of any point  $x \in M_{\mathcal{T}}^3$  in the interior of a  $k$ -cell of  $M_{\mathcal{T}}^3$ ,  $k = 0, 1, 2, 3$ . Heuristically the metric for  $M_{\mathcal{T}}^3$  should be CAT(0), since it is already so for the hyperbolic metric on  $M^3$ , where all links are then standard 2-spheres. The solid angles at 0-cells created by intersecting with  $K_{\mathcal{T}}^h$  are enlarged in  $M_{\mathcal{T}}^3$ , becoming hemispheres: this should not create shorter geodesics. We describe the link structure with a little more care, since the more delicate structure of links in  $K_{\mathcal{T}}^e$  is also revealed. It is important to note that the metric 2-complex  $K_{\mathcal{T}}^h$  is not CAT(0), since the links of 0-cells are not large.

For  $k = 3$ ,  $x$  is an interior point of a Euclidean half space, and its link is thus a standard round sphere,



with all geodesics of length  $2\pi$ . For  $k = 2$ ,  $x$  lies in the interior of some Euclidean polygon  $P_{i,j}^e$ , and its link in  $M_{\mathcal{T}}^3$  is a union of two hemispheres corresponding to the two half-spaces identified along  $P_{i,j}^e$ , and again is a standard sphere. Similarly, the link of  $x \in K_{\mathcal{T}}^e$  is a standard round circle, which is thus large. The piecewise Euclidean metrics are non-singular at points where  $k = 2, 3$ .

When  $k = 1$ , recall that metric singularities of  $M_{\mathcal{T}}^3$  are concentrated on the 1-skeleton of  $K_{\mathcal{T}}^e$ : there are nonsingular vertical edges in each Euclidean upper half space, with trivially large links. Such edges do not lie in  $K_{\mathcal{T}}^e$ . For edges of polygons  $P_{i,j}^e$ , the link is a 2-sphere with two antipodal distinguished points corresponding to the directions along the edge, connected by  $d$  spherical geodesic arcs of length  $\pi$ , where  $d$  is the degree of the edge in  $K_{\mathcal{T}}^e$ . These arcs divide the sphere into  $d$  2-gons, each having the spherical geometry of a hemisphere. The CAT(1) condition is trivially satisfied, since  $d \geq 3$ . Considered as a point in  $K_{\mathcal{T}}^e$ ,  $x$  has link which is a discrete set of  $d$  points, and so trivially large.

The potentially non-CAT(0) links are for  $x$  a 0-cell. In the following, we assume  $n = 3$ , but use  $n$  to indicate how our construction yields CAT(0) metrics in higher dimension. The link of  $x$  in  $M_{\mathcal{T}}^n$  is a union of  $(n - 1)$ -dimensional spherical hemispheres  $Hem_{x,y}^{n-1}$ , one for each horotorus  $H_y$  incident at  $x$ . The equatorial sphere (circle) of each hemisphere is a unit sphere  $S_{x,y}^{n-2}$ , and is a union of spherical polyhedra  $S_{x,y,j}^{n-2}$ , each the link of the vertex  $x$  in a Euclidean polyhedron  $P_{y,j}^e$ . These are circular arcs when  $n = 3$ , with length equal to the angle at a vertex of  $P_{y,j}^e$  incident at  $x$ , and which add to  $2\pi$ , giving  $S^1$  metrically. Since each  $P_{y,j}^e$  is uniquely identified with another  $P_{y',j'}^e$ , the link of  $x \in K_{\mathcal{T}}^e$  is obtained by identifying the spheres  $S_{x,y}^{n-2}$  along corresponding spherical polyhedra  $S_{x,y,j}^{n-2}, S_{x,y',j'}^{n-2}$  (arcs when  $n = 3$ , giving a graph).

Rivin shows that the polar dual of an ideal convex hyperbolic polyhedron in 3-space admits a piecewise spherical geometry obtained by gluing together spherical hemispheres along arcs in their boundary circles. This gives a topological 2-sphere containing an embedded graph whose complementary regions are metric hemispheres. All geodesic loops on the graph have lengths at least  $2\pi$ : more formally [22], for each convex ideal polyhedron  $X$  in  $\mathbb{H}^3$ , let  $X^*$  denote the Poincaré dual of  $X$ . Assign to each edge  $e^*$  of  $X^*$  the weight  $w(e^*)$  equal to the exterior dihedral angle at the corresponding edge  $e$  of  $X$ .

**Theorem 3.** (Rivin [34]). *The dual polyhedron  $X^*$  of a convex ideal polyhedron  $X$  in  $\mathbb{H}^3$  satisfies the following conditions:*

**Condition 1.**  $0 < w(e^*) < \pi$  for all edges  $e^*$  of  $X^*$ .

**Condition 2.** *If the edges  $e_i^*, e_2^*, \dots, e_k^*$  form the boundary of a face of  $X^*$ , then  $w(e_1^*) + w(e_2^*) + \dots + w(e_k^*) = 2\pi$ .*

**Condition 3.** *If  $e_i^*, e_2^*, \dots, e_k^*$  form a simple circuit which does not bound a face of  $X^*$ , then  $w(e_1^*) + w(e_2^*) + \dots + w(e_k^*) > 2\pi$ .*

This result suffices to prove that  $K_{\mathcal{T}}^e$ , and hence  $M_{\mathcal{T}}^3$ , is metrically CAT(0). It remains to reconcile the notions of Poincaré duality and polar duality. We recall the following from [22, 33, 34, 12] in the notation of the latter: Let  $X$  denote a convex polyhedron in  $\mathbb{H}^3$ , viewed in the hyperboloid model in Minkowski space.

**Definition.** The *polar dual*  $P(X)$  for  $X$  is the set of outward-pointing unit normal vectors to the supporting hyperplanes of  $X$ .

For a compact polyhedron, each vertex  $v$  of  $X$  contributes a spherical polyhedron  $lk(v)^*$  to  $P(X)$ : the intrinsic metric on the polar dual  $P(X)$  is obtained by gluing together the spherical polyhedra  $lk(v)^*$  dual to the vertices  $v$  of  $X$ , by isometrics of their edges in the combinatorial pattern described above. For ideal polyhedra, such spherical polyhedra are missing from  $P(X)$ , which is now a piecewise spherical  $(n - 2)$ -complex with distinguished cycles corresponding to the boundaries of missing  $(n - 1)$ -cells. These are the analogues of edges in Condition 2 of Rivin's characterization above.

$P(X)$  inherits the structure of a piecewise spherical cell complex as a subset of the de Sitter sphere in Minkowski space. Each face of  $X$  contributes a spherical cell for  $P(X)$ . Rivin proved in his thesis that the

polar dual of an ideal convex polytope in hyperbolic 3-space is large: Charney and Davis [12] proved the analogous result in higher dimension. The results of [12] are more general than what is required here: our interest is in a strict generalization of Rivin's, where  $X$  arises as a finite volume ideal polyhedron in the Epstein–Penner construction. The pertinent results of [12] are Theorem 4.1.1 and Corollary 4.2.3, which we combine as:

**Theorem 4.** (Charney–Davis). *Suppose  $X$  is a hyperbolic polyhedral set of dimension  $n$ . Then:*

1. *its polar dual  $P(X)$  is large;*
2. *if  $\gamma$  is any closed local geodesic of length  $2\pi$ , then  $\gamma$  must lie in the subcomplex  $P_\gamma$  for some cusp point  $y$  of  $X$ .*
3. *its completed polar dual  $\hat{P}(X)$  is large.*

We will explain this notation shortly: that both  $M_{\mathcal{T}}^3$  and  $K_{\mathcal{T}}^e$  are CAT(0) is then a consequence of:

**Theorem 5.** *The link of the 0-cell  $x$  in  $K_{\mathcal{T}}^e$  is the polar dual  $P(X)$  of the corresponding dual ideal polyhedron  $X$  in the Epstein–Penner canonical decomposition. The link of a 0-cell  $x$  in  $M_{\mathcal{T}}^3$  is the completed polar dual  $\hat{P}(X)$*

*Proof:* Consider the set  $Y$  of parabolic fixed points whose horospheres  $\{H_y\} \subset \mathcal{H}$  meet to define the 0-cell  $x$ . Then  $Y$  is the set of ideal points (cusp points) for the Epstein–Penner canonical ideal polyhedron  $X$  dual to the 0-cell  $x \in K_{\mathcal{H}}$ .

**Definition.** For  $y \in Y$ , let  $E_y$  denote the intersection of  $X$  with a small horosphere at  $y$ , and let  $P_y$  denote its Euclidean polar dual.

Charney–Davis prove that  $P_y$  is locally convex in  $P(X)$ . In our case,  $E_y = P_{i,j}^e$  for some  $i, j$ , since the parabolic fixed point  $y$  corresponds to  $\infty_i$  for some  $i$ . The polar dual of a compact convex Euclidean polyhedron is geometrically a unit sphere, subdivided into spherical sub-polyhedra. In dimension 2, our situation, the polar dual of a compact convex Euclidean polygon is geometrically the unit circle, subdivided into arcs corresponding to the vertices of the polygon, which measure the external ‘turning angles’. In the upper half space model  $\mathbb{U}\mathbb{H}_i^3$ , the faces of the ideal polyhedron  $X$  meeting  $\infty_i$  are vertical, and their normal directions are horizontal and normal to the Euclidean edges of the polygon  $P_{i,j}^e$  obtained by intersecting  $X$  with a horosphere  $HoP_a$  for  $a = 1$ , which is a small enough horosphere. The collection of polar duals  $P_y$  assemble to create  $P(X)$ :

**Lemma 1.** ([12], 2.5.2) *If  $y$  is a cusp point, the subcomplex  $P_y$  of  $P(X)$  corresponding to a cusp is isometric to the polar dual of a convex set  $E_y$  in  $E^{n-1}$ .*

All vertices of  $X$  in the Epstein–Penner construction are ideal vertices, and so all faces of  $X$ , and hence all normals to faces of  $X$ , feature in some subcomplex  $P_y$ . Thus  $P(X)$  is obtained from the disjoint union  $\cup_{y \in Y} P_y$ . In the upper half space model,  $E_y$  is the horizontal slice through a cusp end of  $X$ . These Euclidean polyhedra produce a tessellation of  $HoP_1$ , which is geometrically dual to the tessellation by  $P_{i,j}^e$ . Thus the Poincaré dual to the polyhedron  $X$  yields a Poincaré dual to the polyhedron  $E_y$  as the ‘boundary’ of  $lk(y)$ : the corresponding Poincaré dual cell decomposition of  $\partial E_y$  is combinatorially identical to the link of  $x$  in  $K_{\mathcal{H}}^e \cap HoP_1$ . Geometrically, each vertex  $v$  of  $E_y$  contributes a spherical cell to  $P_y$ , and a spherical cell to the link of  $x$ , and these are identical. These spherical cells of  $P_y$  – circular arcs in the case of  $P_y$  a circle – are identified pairwise in constructing both the link of  $x$  in  $K_{\mathcal{H}}^e \cap HoP_1$ , and  $P(X)$ . It is now a simple matter to complete the picture of the link of  $x$  in  $M_{\mathcal{T}}^3$ :

**Definition.** ([12], 4.2.2). Let  $\text{Cone}(P_y)$  denote the orthogonal join of  $P_y$  with a point. The *completed polar dual* of  $X$ , denoted  $\hat{P}(X)$ , is the piecewise spherical complex formed by gluing  $\text{Cone}(P_y)$  to  $P(X)$  along  $P_y$  for each cusp point  $y$ .

In our case,  $X$  has finite volume, with each  $P_y$  isometric to the round sphere  $S^{n-2}$ , and so  $\text{Cone}(P_y)$  is a hemisphere. Thus,  $\hat{P}(X)$  is obtained from  $P(X)$  by ‘capping off’ each  $P_y$  with hemispheres. It is then clear that  $\hat{P}(X)$  is homeomorphic to the  $(n-1)$ -sphere, and moreover geometrically gives the link of  $x$  in  $M_{\mathcal{T}}^3$ . Corollaries 4.3.1 and 4.2.3 of [12] respectively assert that  $P(X)$  is large, and  $\hat{P}(X)$  is large. Accordingly, the techniques of this paper, and the results of Epstein–Penner, Rivin, and Charney–Davis, show that all non-compact finite volume hyperbolic  $n$ -manifolds admit CAT(0) structures with universal cover obtained as a union of half-spaces, and have CAT(0) spines.

## 9 Canonical deformation from hyperbolic to CAT(0) metrics

In this section we describe how hyperbolic polyhedra admit canonical metric deformations through hyperbolic polyhedra of constant curvature, limiting on a Euclidean structure. We thank Norman Wildberger for a helpful remark on an earlier version of this section, and accordingly define:

**Definition.** The *Wildberger transformation*  $W_\tau : \mathbb{U}\mathbb{H}^n \longrightarrow \mathbb{U}\mathbb{H}^n$ ,  $\tau \geq 0$ , is defined by the formula

$$W_\tau((\mathbf{x}, x_n)) := (\mathbf{x}, \sqrt{x_n^2 + \tau^2}).$$

Thus  $W_0$  is the identity, and  $W_\tau(\mathbb{U}\mathbb{H}^n) = \{(x, x_n) \mid x_n > \tau\}$ , the region above the horosphere  $HoP_\tau$ . The transformation  $W_\tau$  sends horizontal horospheres to horizontal horosphere, and commutes with vertical projection from  $\infty$ . Wildberger transformations are not isometries, but they preserve the collection of piecewise geodesic subsets:

**Proposition 4.** *If  $\mathcal{C}$  is a geodesic arc in  $\mathbb{U}\mathbb{H}^n$ , so is  $W_\tau(\mathcal{C})$ . The hyperplane  $HyS_{y,h}$  is mapped by  $W_\tau$  injectively to an open disc in the hyperplane  $HyS_{y,h}$ .*

*Proof:* This is a simple calculation using Pythagorus’ theorem: it suffices to prove this for the hyperbolic plane (the case  $n = 2$ ), since any geodesic arc with endpoints in  $\mathbb{R}_\infty^{n-1}$  lies in a vertical plane. This is illustrated in Figure 2, with two given geodesic arcs  $P_0Q_0$ ,  $Q_0R_0$ . Suppose  $P_0 = (X, a_0)$ ,  $Q_0 = (Y, b_0)$ ,  $R_0 = (Z, c_0) \in \mathbb{U}\mathbb{H}^2$  are three arbitrary points, with  $P_0Q_0$  and  $Q_0R_0$  hyperbolic geodesic arcs, manifest as arcs of semicircles centred at points on  $\mathbb{R}_0^1 := \{(\mathbf{x}, 0)\} \subset \mathbb{R}^2$ . Let  $|XY| = u$ ,  $|YZ| = v$ . With respect to the Euclidean metric on  $\mathbb{R}^2$ ,  $|XQ_0| = a_0$ ,  $|Q_0Z| = c_0$ . Thus

$$\begin{aligned} u^2 + b_0^2 &= a_0^2, & v^2 + b_0^2 &= c_0^2 \\ \implies u^2 + (b_0^2 + \tau^2) &= (a_0^2 + \tau^2), & v^2 + (b_0^2 + \tau^2) &= (c_0^2 + \tau^2) \quad \forall t \in \mathbb{R} \end{aligned} \tag{1}$$

Setting  $P_\tau = W_\tau(P_0) = (X, a_\tau)$ ,  $Q_\tau = W_\tau(Q_0) = (Y, b_\tau)$ ,  $R_\tau = W_\tau(R_0) = (Z, c_\tau)$ , we see that Equation 1 is equivalent to

$$u^2 + b_\tau^2 = a_\tau^2, \quad v^2 + b_\tau^2 = c_\tau^2,$$

and so all points of the hyperbolic geodesic arcs  $P_0Q_0$ ,  $Q_0R_0$  move vertically to corresponding points on the hyperbolic geodesic arcs  $P_\tau Q_\tau$ ,  $Q_\tau R_\tau$ , which also appear as arcs of Euclidean circles:  $W_\tau(P_0Q_0) = P_\tau Q_\tau$  as a transformation of hyperbolic geodesic arcs, as claimed. Every point on an arc moves vertically under  $W_\tau$  towards  $\infty$ , by a distance depending only on its initial height.

**Corollary 1.** *The image  $W_\tau(C)(\Pi)$  of any  $k_i$ -dimensional hyperbolic hyperplane  $\Pi$  is an open ball in some  $k_i$ -dimensional hyperbolic hyperplane  $\Pi'$ . If  $C \subset \mathbb{U}\mathbb{H}^n$  is contained in some  $k_i$ -dimensional hyperbolic hyperplane, so is  $W_\tau(C)$ .*

We now examine how the hyperbolic geometry is distorted. Consider the angles  $\angle P_0Q_0R_0$ ,  $\angle P_\tau Q_\tau R_\tau$  between the two arcs at  $Q_0$ ,  $Q_\tau$ . From basic geometry we have:



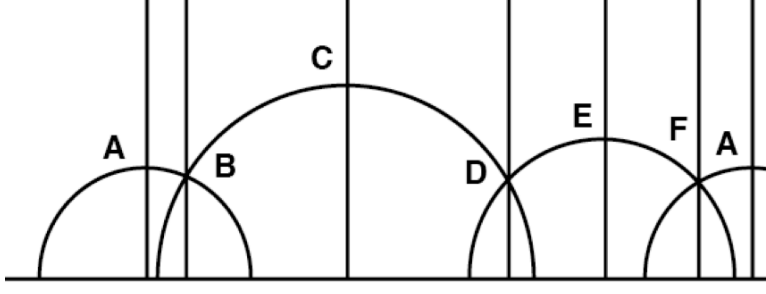


Figure 3: In the limit, Wildberger transformations applied to a closed neighbourhood of a cusp of a Riemann surface, with piecewise geodesic boundary, give a Euclidean product structure isometric to  $S^1 \times [1, \infty)$ .

combinatorial identifications continue to be geometrically feasible by isometry, and equivariantly with respect to group actions: this defines a new metric on  $M^n$  for each  $\tau, \mathcal{T}$ .

By rescaling the curvature by  $\tau$  we obtain a metric deformation through piecewise constant curvature metrics with limit the Euclidean metric  $M^n_{\mathcal{T}}$ .

## 10 Weighted points: reconstitution of geometric structure

Given a pair  $M^3, \mathcal{T}$ , we have constructed a polyhedral decomposition, with polyhedra in  $K^h_{\mathcal{T}}$  assigned the label identifying hyperplanes in which they lie in  $\mathbb{UH}^3$ . These hyperplanes arise as the intersection of a descending horosphere  $HoP_{e^{-t}}$  and an expanding horosphere  $HoS_{*,het}$ . All polygons  $P^e_{i,j}$  arise from the collision locus of expanding Euclidean circles, as viewed from infinity projected to  $HoP_1$ . Thus knowing the initial moment of birth of each circle, and its location, the geometric and combinatorial data can be reconstructed for both the hyperbolic or Euclidean structure: the Ford balls can be reconstructed, as can the combinatorial structure of their boundaries. In this section, we merely record the nature of expansion of circles corresponding to a labeling of polygons:

**Lemma 3.** *Suppose the hyperbolic plane  $HyS_{p,h}$  contains a polygon labeled  $(p,h)$ . Let  $C_{p,h}(t)$  denote the projection to  $HoP_1$  of the intersection  $S^1_{pq}(t)$  of  $HyS_{p,h}$  with  $HoP_{e^{-t}}$ . Then the radius  $r_{p,h}(t)$  of this circle satisfies*

$$r^2_{p,h}(t) = e^{-2t_0}(1 - e^{-2(t-t_0)}), \quad t \geq t_0.$$

*Proof:* Again, it suffices to work in  $\mathbb{UH}^2$ : we assume  $p = 0$ , and  $h = e^{-t_0}$ . The hyperplane is ‘born’ at  $t = t_0$  as the plane  $HoP_1$  descends at unit speed, starting when  $t = 0$ . Parametrize the semi-circle  $HyS_{0,e^{-t_0}}$  by  $x = e^{-t_0} \tanh u$ ,  $y = e^{-t_0} \operatorname{sech} u = e^{-t}$ . Simple algebra gives the result.

Hence the circle expansion slows exponentially quickly. Given an arbitrary finite set of weighted points in the plane, and a lattice  $\mathcal{L} \cong \mathbb{Z} \oplus \mathbb{Z}$ , we can attempt to create a tessellation of the corresponding elliptic curve. One circle may be created in the interior of another expanding circle at a later time; moreover, it may fail to expand to meet the larger one, or overtake to create an edge as expansion continues. However, the data given by the creation of  $K^e_{\mathcal{T}}$  ensures a true tessellation by compact polygons occurs, and  $K^h_{\mathcal{T}}$  can be constructed, allowing the hyperbolic geometry of each Ford ball to be realized.

**Theorem 7.** *Given the weights assigned to polygons  $P^e_{i,j}$ , we can reconstruct the hyperbolic metric of  $M^3$ .*

There are additional properties of such weighted-point sets among arbitrary ones, related to Pythagorus’ equation.

## 11 Snappea, SnapPy: very snappy

In the cusped case, the canonical deformations described above can be applied simultaneously to all fundamental regions of hyperbolic space itself: the universal cover of a flattened manifold is a flattening of hyperbolic space, and hence offers a model for hyperbolic space. We see a union of half spaces, each with boundary plane biperiodically decomposed as a union of Euclidean polygons. These polygons are pairwise-identified by Euclidean isometry, and we can therefore imagine navigating in the complement of the singular 1-skeleton by usual motion in Euclidean space. We can consider developing maps into Euclidean space; interesting number theoretic questions arise concerning the Euclidean translations and rotations so obtained.

Both software packages Snappea and SnapPy [40, 14] allow the user to see the Ford domains for cusped hyperbolic 3-manifolds, and interactively adjust the defining cusp tori. Thus all of the CAT(0) structure described in this paper is in principle visible in this way. However, the interface does not provide independent windows for the simultaneous viewing of normalized upper half space models: this would be a valuable addition.

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